J. Fluid Mech. (1970), *vol.* 42, *part* 1, *pp.* 207–217
Printed in Great Britain

The oseenlet as a model for separated flow in a rotating viscous liquid

By JOHN W. MILES

Institute of Geophysics and Planetary Physics and Department of Aerospace and Mechanical Engineering Sciences, University of California, La Jolla

(Received 4 August 1969)

The disturbance induced in the uniform flow of a viscous, rotating liquid by an axial point force $-D$ is studied under the restrictions that the Ekman number, $E = 2\Omega\nu/U^2$, be small and that $D = O(1/\log E)$ as $E \to 0$. The method of matched asymptotic expansions is invoked to obtain inner and outer (with reference to the dimensionless axial co-ordinate x refered to the length $U/(2\Omega)$) approximations to the solution of the Oseen equations as $E \to 0$. The outer approximation, $E \to 0$ with Ex fixed, is also an outer approximation to the solution of the Navier–Stokes equations. The mass flow across any transverse plane, which is equal to D/U for an oseenlet in a non-rotating flow, vanishes in this approximation. The corresponding inner limit yields a non-uniform, cylindrical flow far upstream of the force in the inviscid limit, $E \to 0$, if and only if $D \propto 1/(\log E + \text{const.})$. This cylindrical flow is a one-term, inner approximation to the solution of the Navier–Stokes equations and suffices to show that separation implies the failure of Long's hypothesis of no upstream influence for inviscid, rotating flow past a finite body. A two-term inner representation of the solution is related to Stewartson's solution of the Oseen equations for a moving source in an inviscid, rotating fluid.

1. Introduction

We consider the disturbance induced in the uniform flow of a slightly viscous, rotating liquid by a point force of magnitude D directed along the upstream axis. Let U, Ω, ν and ρ denote the translational and angular velocities of the basic flow and the kinematic viscosity and density of the liquid. Choosing

$$L = U/(2\Omega) \tag{1.1}$$

as a characteristic length, we construct the Ekman number

$$E = 2\Omega\nu/U^2 = \nu/(2\Omega L^2) = \nu/(UL) \tag{1.2}$$

and the perturbation amplitude

$$\epsilon = D/(4\pi\rho U^2 L^2) \tag{1.3}$$

and seek inner and outer approximations to the asymptotic solution for the disturbance on the hypothesis (the necessity for which is developed below)

$$\epsilon = O(1/\log E) \quad (E \to 0). \tag{1.4}$$

The flow at sufficiently large distances from a body in a non-rotating, viscous liquid is governed by the Oseen equations (to which the Navier–Stokes equations reduce on the neglect of terms of second order in the perturbation velocity). The fundamental solution of the Oseen equations for a point force is known as an *oseenlet* (Van Dyke 1964, p. 158) and exhibits a source-like behaviour except in a paraboloidal wake, $\tilde{x} > 0$ and $\tilde{r}^2 = O(\nu\tilde{x}/U)$ in the cylindrical co-ordinates $\tilde{x}$ and $\tilde{r}$. The total mass efflux across a surface that surrounds the singular point, but from which the wake cross-section is excised, is D/U; this efflux is balanced by a corresponding influx within the wake.

The Oseen equations for the rotating flow described in the opening paragraph may be placed in the form

$$(\mathbf{U}.\nabla)\,\mathbf{v}+2\mathbf{\Omega}\times(\mathbf{v}-\mathbf{\Omega}\times\mathbf{r}) = -\rho^{-1}\nabla p-\nu\nabla\times\nabla\times\mathbf{v}+\mathbf{F}, \quad \nabla.\mathbf{v}=0, \qquad (1.5a,b)$$

where $\mathbf{U}$ and $\mathbf{\Omega}$ are directed along the x axis, $\mathbf{v}$ is the particle velocity, p is the reduced pressure (including centrifugal pressure), and $\mathbf{F}$ is the body force. Childress (1964) gives a solution of (1.5) for an oseenlet in a very viscous fluid on the hypothesis that $E = O(1)$ as $U\mathbf{r}/\nu \to \infty$. This solution is directly relevant in the present context only for $E\mathbf{r}/L \to \infty$, but it does imply the striking result, *vis-a-vis* the non-rotating flow, that the mass flow across any transverse plane vanishes. We find that this result also holds for $E \to 0$ with Ex fixed, where, here and subsequently, x and r are dimensionless cylindrical co-ordinates referred to L.

The outer approximation to the solution of (1.5) for the dimensionless perturbation stream function has the form $\epsilon\psi_0(Ex,r)$, where $\psi_0 = O(1)$ as $E \to 0$ with Ex fixed. We find that the inner expansion of $\epsilon\psi_0$, $E \to 0$ with x fixed, is of the form $\epsilon(\log|Ex|+\text{const.})\,rJ_1(r)$; accordingly, the inner limit of $\epsilon\psi_0$ exists if and only if $\epsilon = O(1/\log E)$, as anticipated in (1.4). Proceeding on this hypothesis, we find that the first two terms, of $O(\epsilon\log E)$ and $O(\epsilon)$, in the inner expansion of $\epsilon\psi_0$ can be matched to a two-term inner approximation, $E \to 0$ with x fixed, that satisfies the inviscid Oseen equations, $\nu = 0$ in (1.5). The dominant component of this inner approximation is a non-trivial, inviscid solution if and only if $\epsilon \propto 1/(\log E+\text{const.})$, in which instance it is cylindrical (independent of x) and yields the upstream velocity

$$\mathrm{v}_i \sim U\{1-\mathscr{U}J_0(r), 0, \tfrac{1}{2}r-\mathscr{U}J_1(r)\} \quad [E \to 0, x \to -\infty, r = O(1)], \qquad (1.6)$$

where $\{-,-,-\}$ comprises axial, radial, and azimuthal components. The subdominant component of the inner approximation comprises Stewartson's (1968*a*) solution for a source in an inviscid, rotating flow and (at least partially) resolves the difficulties posed by singularities in that solution.

The outer approximation, $\epsilon\psi_0(Ex,r)$, to the solution of (1.5) under the hypothesis (1.4) is also an outer approximation to the solution of the full Navier–Stokes equations, which differ from (1.5) by terms of $O(\epsilon^2)$. Including these second-order terms in the construction of a two-term outer expansion of $\epsilon\psi_0(Ex,r)$ yields terms of $O(\epsilon^2\log|Ex|) = O(\epsilon)$ in the two-term inner expansion if $\epsilon \propto 1/(\log E+\text{const.})$, in consequence of which only the dominant (cylindrical) term of the inner approximation to the solution of the Oseen equations may be regarded as an inner approximation to the asymptotic solution of the Navier–Stokes equations. This fundamental difficulty (escalation of logarithmic terms) in the method of matched asymptotic expansions, which appears already in the prototype problem of low Reynolds-number flow past a circle (Van Dyke 1964, §§ 8.7, 10.5) and is rather fully explored by Fraenkel (1969), appears to limit the quantitative significance of the Oseen approximation. (On the other hand, Stewartson (1968*b*) gives empirical arguments in support of the Oseen approximation and suggests that it should yield qualitatively valid predictions outside of viscous shear layers.)

This study was originally undertaken in connexion with the question of upstream influence in an axisymmetric, inviscid, rotating flow past a prescribed stream surface, say S. The assumption that such a flow is unseparated yields the upstream velocity (Miles 1969).

$$\mathbf{v}_i \sim U\{1+F_1 x^{-1} J_0(r), 0, \tfrac{1}{2}r + F_1 x^{-1} J_1(r)\} \quad [E \to 0, x \to -\infty, r = O(1)], \tag{1.7}$$

where F_1 is the (dimensionless) *dipole moment* of S. This result depends essentially on the hypothesis that the fluid particles at every point on S originate on the upstream axis (Miles 1970); accordingly, it does not hold for separated flow, in which particles that originate on the upstream axis leave S at a separation ring and proceed downstream along a stream surface that forms the outer boundary of a wake. The theoretical determination of this wake boundary in the neighbourhood of S poses a problem that is likely to prove even more intractable than its counterpart for non-rotating flows; however, the model of an oseenlet does yield a valid description of the separated flow in the far field and does imply that Long's (1953) hypothesis, of no upstream influence for axisymmetric, inviscid, rotating flow past a prescribed body, fails for separated flow in the sense that the upstream flow described by (1.6) in the ordered limit $E \to 0$ and $x \to -\infty$ is *not* uniform.† This conclusion, which contrasts with the opposite conclusion implied by (1.7) for unseparated flow, is in agreement with that advanced by Stewartson (1968*c*), who suggests that "the unsatisfactory properties of (his) elementary source solution...suppl[y] further evidence that if [a] body [in an unbounded, rotating flow] is experiencing drag...an upstream wake will occur for all $[a/L] > 0$ where a is a characteristic diameter of the body." (It is not clear from this statement whether Stewartson's 'drag' includes wave drag, which, in contrast to viscous drag, depends essentially on the non-linear terms in the equations of motion and therefore cannot be properly described by the Oseen approximation.)

We emphasize that the limiting flow (1.6) is primarily of theoretical interest in connexion with the formal question of upstream influence in an inviscid flow; it is quite unlikely to be of quantitative significance for the parametric régimes that can be realized in laboratory configurations. Indeed, the comparison (Miles 1969) of the inviscid approximation (1.7) for the axial velocity upstream of an ellipsoid with Maxworthy's (1970) measured values upstream of the approxi-

† It may be of some interest to observe that choosing $\mathscr{U} = 1$ in (1.6) yields a cylindrical flow that is brought to rest on the axis, thereby resembling a Taylor column.

mately ellipsoidal, forward wake of a sphere suggests that there exists a significant domain in which the inviscid calculation on the hypothesis of a closed stream surface provides quantitatively significant results.

2. Equations of motion

We choose L and U as scales of length and velocity and pose the position, velocity, and vorticity vectors in the forms

$$\mathbf{r} = L\{x, r, 0\} = L\{R\cos\theta, R\sin\theta, 0\}, \tag{2.1}$$

$$\mathbf{v} = U\{1, 0, \tfrac{1}{2}r\} + Ur^{-1}\{\psi_r, -\psi_x, \gamma\}, \tag{2.2}$$

and

$$\boldsymbol{\omega} = 2\Omega\{1, 0, 0\} + 2\Omega r^{-1}\{\gamma_r, -\gamma_x, \chi\} \equiv \nabla\times\mathbf{v}, \tag{2.3}$$

where the triad $\{-,-,-\}$ comprises the axial, radial, and azimuthal components of a vector, subscripts imply partial differentiation, ψ is a perturbation stream function, γ is the azimuthal circulation relative to the basic flow, and χ/r is the azimuthal vorticity. We also introduce the column matrix ($\boldsymbol{\psi}$ is *not* a vector in the polar co-ordinate space)

$$\boldsymbol{\psi}(x, r) = \{\psi, \gamma, \chi\}. \tag{2.4}$$

Substituting (2.1)–(2.3) into (1.5a), choosing

$$\rho\mathbf{F} = -\{D, 0, 0\}\,\delta(Lx)\,\delta(Lr)/(2\pi Lr), \tag{2.5}$$

where δ is Dirac's delta function, eliminating the pressure, and invoking (1.2) and (1.3), we obtain

$$\mathscr{D}\psi = -\chi, \tag{2.6a}$$

$$E\mathscr{D}\gamma = \gamma_x - \psi_x, \tag{2.6b}$$

and

$$E\mathscr{D}\chi = \chi_x - \gamma_x - 2\epsilon r\partial_r r^{-1}\delta(x)\,\delta(r), \tag{2.6c}$$

where

$$\mathscr{D}\psi \equiv (\partial_x^2 + r\partial_r r^{-1}\partial_r)\,\psi \tag{2.7}$$

and

$$\partial_x\psi \equiv \psi_x \equiv (\partial\psi/\partial x). \tag{2.8}$$

We remark that (2.6a) is an exact, kinematical indentity, whereas (2.6b) and (2.6c) differ from the exact results deduced from the Navier–Stokes equations (cf. Goldstein 1938, p. 115) by the omission from their right-hand sides of the second-order terms

$$\Gamma = r^{-1}\partial(\gamma, \psi)/\partial(x, r) \tag{2.9a}$$

and

$$X = r^{-1}\partial(\chi, \psi)/\partial(x, r) + 2r^{-2}(\chi\psi_x - \gamma\gamma_x), \tag{2.9b}$$

respectively.

We seek the solution of (2.6) for $\boldsymbol{\psi}(x, r)$ subject to the boundary conditions

$$\boldsymbol{\psi}(x, 0) = 0 \quad (x \neq 0) \tag{2.10a}$$

and

$$\boldsymbol{\psi}(x, r) \to 0 \quad (R \to \infty), \tag{2.10b}$$

which follow from the requirements that the velocity, vorticity, and perturbation in the total linear and angular momenta of the fluid be bounded.

3. Outer approximation

We proceed on the hypothesis that (2.6) and (2.10) admit an outer approximation of the form

$$\boldsymbol{\psi}^{(0)} \sim \epsilon[\boldsymbol{\psi}_0(\xi, r) + o(1)], \quad \xi \equiv Ex = O(1), \quad Er = o(1), \tag{3.1}$$

where, here and subsequently, the asymptotic approximations and associated order symbols refer to the limit $E \to 0$ (or, more precisely, $E \downarrow 0$). We also may regard (3.1) as an outer solution of the Navier–Stokes equations by virtue of (1.4) and the fact that Γ and X, as given by (2.9), are $O(\epsilon^2)$.

Introducing the change of variable $\xi = Ex$ in (2.6) and letting $E \to 0$, we obtain

$$\mathscr{D}_0\psi = -\chi, \quad \mathscr{D}_0\gamma = \gamma_\xi - \psi_\xi, \quad \mathscr{D}_0\chi = \chi_\xi - \gamma_\xi - 2\epsilon r\partial_r r^{-1}\delta(\xi)\,\delta(r), \tag{3.2a, b, c}$$

where

$$\mathscr{D}_0 = r\partial_r r^{-1}\partial_r. \tag{3.3}$$

Solving (3.2) and (2.10) with the aid of Fourier and Hankel transformations with respect to x and r, respectively, and placing the result in the form (3.1), we obtain

$$\boldsymbol{\psi}_0 = -r\int_0^\infty \{\beta, -\beta^2, \beta^3\}(1+\beta)^{-1}\exp[-\beta^3(1+\beta)^{-1}\xi]\,J_1(\beta r)\,d\beta$$

$$-r\int_1^\infty \{\beta, \beta^2, \beta^3\}(\beta-1)^{-1}\exp[-\beta^3(\beta-1)^{-1}\xi]\,J_1(\beta r)\,d\beta \quad (\xi > 0) \tag{3.4a}$$

$$= -r\int_0^1 \{\beta, \beta^2, \beta^3\}(1-\beta)^{-1}\exp[-\beta^3(1-\beta)^{-1}|\xi|]\,J_1(\beta r)\,d\beta \quad (\xi < 0). \tag{3.4b}$$

We infer from (3.4) that $\psi_0 \to 0$ for $r \to 0$ or $r \to \infty$ (introduce the change of variable $t = \beta r$ and let $r \to \infty$), in consequence of which the net mass flux across any transverse plane (x = const.) vanishes for $x = O(1/E)$.† This paradox, *vis-a-vis* the result for non-rotating flow, in which the mass flux across any transverse plane is equal to D/U, reflects the decisive role of the Coriolis force for $x = O(1/E)$, a régime that has no counterpart in a non-rotating flow.

We obtain the dominant terms in the inner expansion of $\boldsymbol{\psi}_0$, $E \to 0$ with x fixed, by separating out the singular components of the integrands at $\beta = 1$, introducing the change of variable $t = 1/(1-\beta)$ in these components, and then letting $\xi \to 0$. Letting $\xi \uparrow 0$ in (3.4b), we obtain

$$\boldsymbol{\psi}_0(\xi < 0) \sim \mathbf{I}(\log|\xi| + C)\,rJ_1(r) + \hat{\boldsymbol{\psi}}(r) \quad (\xi \uparrow 0), \tag{3.5}$$

where $\mathbf{I} \equiv \{1, 1, 1\}$, C is Euler's constant,

$$\hat{\boldsymbol{\psi}}(r) = \mathbf{I}\hat{\psi}(r) + \left\{0, -J_0(r) + r^{-1}\int_0^r J_0(t)\,dt, 2r^{-1}J_1(r) - 2J_0(r) + r^{-1}\int_0^r J_0(t)\,dt\right\}, \tag{3.6}$$

† Strictly speaking, the outer approximation is valid only for $r = o(1/E)$, and the mass flux outside of this domain should be estimated separately. A heuristic estimate, based on the asymptotic behaviour of ψ_0 as $r \to \infty$, implies that the mass flux across x = const., $r > r_1$ is $O(r_1^{-\frac{1}{2}})$.

and
$$\hat{\psi}(r) = r\int_0^1 (1-\beta)^{-1}[J_1(r)-\beta J_1(\beta r)]\,d\beta \tag{3.7a}$$
$$= \tfrac{1}{2}r^2(\tfrac{3}{2}-\tfrac{25}{96}r^2+\dots). \tag{3.7b}$$

We find it expedient, in determining the inner expansion of ψ_0 for $\xi\downarrow 0$, to rewrite (3.4a) in the form
$$\psi_0 \equiv \psi_0(\xi<0)+\mathscr{H}(\xi)\,\phi(\xi,r), \tag{3.8}$$
where $\psi_0(\xi<0)$ is given by (3.4b), $\mathscr{H}(\xi)$ is Heaviside's step function,
$$\phi(\xi,r) = \phi_+(\xi,r)+\phi_-(\xi,r), \tag{3.9a}$$
and
$$\phi_\pm(\xi,r) = -r\int_0^\infty \{\beta,\ \mp\beta^2, \beta^3\}(\beta\pm 1)^{-1}\exp(-\beta^3|\beta\pm 1|^{-1}\xi)\,J_1(\beta r)\,d\beta. \tag{3.9b}$$

The inner expansion of ϕ as $\xi\downarrow 0$ is complicated by the existence of a viscous wake for $r^2 = O(\xi)$, $\xi>0$, just as in the corresponding problem for a non-rotating fluid. Introducing the wake co-ordinate (which is independent of Ω)
$$\zeta = \tfrac{1}{4}(Ex)^{-1}r^2 \equiv \tfrac{1}{4}U(\nu\tilde{x})^{-1}\tilde{r}^2 \tag{3.10}$$
and letting $\xi\downarrow 0$, we obtain (see appendix for details)
$$\begin{aligned}\phi(\xi,r) &\sim \mathbf{I}[(1-2\xi)\,\pi rY_1(r)+2(1-3\xi-\tfrac{1}{8}r^2)\,e^{-\zeta}+\tfrac{1}{2}r^2E_1(\zeta)+O(\xi^2)]\\ &\qquad +2\zeta e^{-\zeta}\{0,1+O(\xi),\ -\xi^{-1}+1+\tfrac{1}{2}\zeta+O(\xi)\}\quad (\xi\downarrow 0) &(3.11a)\\ &\sim \pi\mathbf{I}rY_1(r)\quad(\xi\downarrow 0,\zeta\to\infty) &(3.11b)\\ &\sim \mathbf{I}[-2\xi+\tfrac{1}{2}r^2(-\xi^{-1}+\log\xi+C+\tfrac{3}{2})]+2\zeta\{0,1,\ -\xi^{-1}+1\}\\ &\qquad (\xi\downarrow 0,\zeta\downarrow 0), &(3.11c)\end{aligned}$$
where
$$E_1(\zeta) = \int_\zeta^\infty e^{-t}t^{-1}dt = -\log\zeta - C+\int_0^\zeta (1-e^{-t})\,t^{-1}\,dt \tag{3.12}$$
is the exponential integral.

Substituting (3.5) and (3.11b, c) into (3.8) we obtain the two-term inner expansions [comprising terms of $O(\epsilon)$ and $O(\epsilon\log E)$]
$$\psi^{(0)} \sim \epsilon\big(\mathbf{I}[(\log|Ex|+C)\,rJ_1(r)+\pi\mathscr{H}(x)\,rY_1(r)]+\hat{\psi}(r)\big)+o(\epsilon)$$
$$[x = O(1), \mathscr{H}(x)\,(Ex)^{\frac{1}{2}} \ll r \ll E^{-1}] \tag{3.13}$$
for points outside of the wake and
$$\psi^{(0)} \sim \epsilon\big(\mathbf{I}r^2[-\tfrac{1}{2}(Ex)^{-1}+\log(Ex)+C+\tfrac{3}{2}]+2(Ex)^{-1}r^2\{0,1,\ -(Ex)^{-1}+1\}\big)$$
$$+o(\epsilon)\quad [x = O(1),\ r^2 \ll Ex] \tag{3.14}$$
for points near the axis of the wake.

We infer from (3.13) that the inner limit of $\psi^{(0)}$ exists if and only if $\epsilon = O(1/\log E)$; it differs from zero if and only if
$$\epsilon = -\mathscr{U}(\log E+A)^{-1}, \tag{3.15}$$
where $\mathscr{U}$ and A are constants ($\mathscr{U}>0$ implies $\epsilon>0$ as $E\to 0$). Invoking (3.15) in (3.13), we obtain the inner limit
$$\psi^{(0)} \sim -\mathbf{I}\mathscr{U}rJ_1(r)+O(1/\log E)\quad [x = O(1),\ 0<r = O(1)]. \tag{3.16}$$

The approximations (3.13) and (3.14) suggest the existence of axial stagnation points within the inner domain of the outer approximation. The corresponding approximations to the perturbation velocity on the axis are

$$u_0 \equiv [r^{-1}\psi_r^{(0)}]_{r=0} \tag{3.17a}$$
$$\sim \epsilon(\log|Ex| + C + \tfrac{3}{2}) + o(\epsilon) \quad (1 \ll -x \ll E^{-1}), \tag{3.17b}$$
$$\sim \epsilon[-(Ex)^{-1} + 2(\log Ex + C + \tfrac{3}{2})] + o(\epsilon) \quad (1 \ll x \ll E^{-1}). \tag{3.17c}$$

Setting $x = x_0 < 0$ and $u_0 = -1$ in (3.17b), we obtain

$$x_0 = -E^{-1}\exp(-\epsilon^{-1} - C - \tfrac{3}{2}) \tag{3.18}$$

for the upstream stagnation point. This is within the assumed domain, $E|x_0| \ll 1$, if and only if (within the restriction $\epsilon = O(1/\log E)$) ϵ is given by (3.15). The downstream stagnation point implied by (3.17c) is within the assumed domain for all $\epsilon = o(1)$; it tends to infinity as $E \to 0$ if and only if $E/\epsilon = o(1)$.

4. Inner approximation

We seek that inner solution of (2.6) and (2.10), say $\boldsymbol{\psi}^{(i)}$, which matches the two-term outer representation (3.13) on the hypothesis that $\epsilon = O(1/\log E)$.

Letting $E \to 0$ in (2.6), we obtain

$$\mathscr{D}\psi + \chi = 0 \quad \text{and} \quad \psi_x = \gamma_x = \chi_x \quad (R \neq 0) \tag{4.1a, b}$$

with a relative error of $O(E)$. Excluding the wake, $r^2 = O(Ex)$ and $x > 0$, from the inner domain and invoking the known results for the corresponding solution in a non-rotating fluid, we infer that $\boldsymbol{\psi}^{(i)}$ must exhibit a source-like behaviour in the neighbourhood of the singular point, $R = 0$, and must yield a total mass flux of D/U, or 4π in dimensionless terms, through any closed surface that includes $R = 0$. Choosing $\psi = 0$ on the upstream axis, we infer from this mass-flux requirement that

$$\psi^{(i)}(x, 0) = -2\epsilon\mathscr{H}(x). \tag{4.2}$$

Guided by the form of (3.13) and the fact that both $\mathbf{I}rJ_1(r)$ and $\hat{\boldsymbol{\psi}}(r)$ are solutions of (4.1), we pose the solution of (4.1) and (4.2) in the form

$$\boldsymbol{\psi}^{(i)} \sim \epsilon\big(\mathbf{I}[(\log E)\, rJ_1(r) + \psi_s(x, r)] + \hat{\boldsymbol{\psi}}(r)\big) + o(\epsilon), \tag{4.3}$$

where $\epsilon\psi_s$ satisfies (4.2) and (for $R \neq 0$)

$$\mathscr{D}\psi + \psi = 0. \tag{4.4}$$

Letting $E \to 0$ in (4.3) with Ex and r fixed and invoking the requirement that $\boldsymbol{\psi}^{(i)}$ match the inner representation (3.13), we obtain the matching requirement

$$\psi_s(x, r) \sim (\log|x| + C)\, rJ_1(r) + \pi\mathscr{H}(x)\, rY_1(r) \quad (|x| \to \infty). \tag{4.5}$$

(It suffices, and is perhaps a more obvious procedure, to invoke this requirement as $x \to -\infty$, after which its satisfaction for $x \to \infty$ follows implicitly from the subsequent, asymptotic representation of ψ_s.)

Stewartson (1968a) obtains a source-like solution of (4.4) by integrating Fraenkel's (1956) dipole solution, say $\psi_d(x, r)$, with respect to x and then determining the concomitant, arbitrary function of r by considering Fraenkel's solution for a source in a cylindrical pipe of radius b in the limit $b \to \infty$. This limit contains a term of the form $f(b)\,rJ_1(r)$, where $f(b)$ has an infinite, discrete set of poles, corresponding to the discrete modes of the pipe, and, in addition, a logarithmic singularity at $b = \infty$; however, as Stewartson (1968*b*) observes, this difficulty may be circumvented by adding an appropriate multiple of the eigensolution $rJ_1(r)$. Determining the coefficient of this eigensolution through the invocation of (4.5), we obtain

$$\psi_s(x,r) = \pi\mathscr{H}(x)\,rY_1(r) + [C + \log|x| - Ci(|x|) + x^{-1}\sin x - x^{-2}(1-\cos x)]\,rJ_1(r) - \operatorname{sgn} x\, r\partial_r \int_R^\infty (t^2 - r^2)^{-\frac{1}{2}} \cos t\, dt - r\int_0^1 [J_1(\beta r) - \beta J_1(r)] \cos[x(1-\beta^2)^{\frac{1}{2}}] \times (1-\beta^2)^{-1}\beta^2\, d\beta, \quad (4.6)$$

where

$$Ci(|x|) = C + \log|x| - \int_0^{|x|} (1-\cos t)\, t^{-1} dt \quad (4.7)$$

is the cosine integral. This representation differs from $2\psi_1$, where ψ_1 is the source solution given by equation (14) in Stewartson's (1968*a*) paper, by a multiple of $rJ_1(r)$, the coefficient of which in (4.6) has been determined by (4.5). We also note the following approximations to ψ_s:

$$\psi_s(x,r) \sim \pi\mathscr{H}(x)\,rY_1(r) + [C + \log|x| + O(x^{-2})]\,rJ_1(r) + 2\mathscr{H}(x)\,(xR)^{-1} r^2 \cos R + O(R^{-1}) \quad (R \to \infty,\ |\theta - \tfrac{1}{2}\pi| > 0), \quad (4.8)$$

in which the term in $\cos R$ represents the lee-wave field of the source;

$$\psi_s(x,r) = -2\mathscr{H}(x) + r^2\mathscr{H}(x)\,[C - \tfrac{1}{2} + \log\tfrac{1}{2}r - Ci(x) + x^{-1}\sin x + x^{-2}\cos x] + \tfrac{1}{2}r^2(C + \log|x| - x^{-2}) + O(r^4) \quad (r \to 0), \quad (4.9)$$

which exhibits the logarithmic singularity in ψ_s as $r \to 0$;

$$\psi_s(x,r) = -(1+\cos\theta) + \tfrac{1}{2}r^2[C + \log\tfrac{1}{2}(R-x)] + O(R^4) \quad (R \to 0), \quad (4.10)$$

in which the first term represents the solution for a source in potential flow.

Stewartson (1968*b*) includes an inviscid source solution of the form

$$\boldsymbol{\psi}_i(x,r) = \mathbf{I}[\psi_s(x,r) + C_0 rJ_1(r)] + \hat{\boldsymbol{\psi}}(r), \quad (4.11)$$

where C_0 is independent of E, as one component of his solution for rotating flow past a sphere. This solution cannot be matched to the outer representation (3.13), even within the context of the Oseen approximation, without violating the restriction $\epsilon = O(1/\log E)$. It is, of course, possible that (4.11) could be replaced by (4.3) in Stewartson's formulation with only minor modifications of his end results. In any event, the present interpretation of ψ_s, as a component of the inner representation (4.3) of the solution of the Oseen equations for an oseenlet, resolves the difficulties posed by the singularities in ψ_s for either $|x| \to \infty$ or $r \to 0$ with $x > 0$. We also note that Stewartson's solution for flow past a sphere of radius La, $a \ll 1$, yields an axial velocity of $U = 0{\cdot}132a^2$ on $r = 0$ as $|x| \to \infty$. This prediction of an accelerated axial flow contrasts with the decelerated flow implied by (3.17*c*).

(Professor Stewartson (private communication) has emphasized that the logarithmic singularity in ψ_s as $|x| \to \infty$ is a consequence of the limit $b \to \infty$. He also suggests that the joint limit $E \to 0$, $b \to \infty$ may not be uniform and that "the difficulty associated with $D = O(1/\log E)$ may well disappear if the more general and more realistic problem (of a tube of finite radius) is discussed.")

We conclude by remarking that the one-term inner approximation obtained by invoking (3.15) in (4.3),

$$\boldsymbol{\psi}^{(i)} \sim -\mathbf{I}\mathscr{U} r J_1(r) + O(1/\log E), \tag{4.12}$$

is identical with the inner limit of (3.16), is a cylindrical-wave solution of (4.4), and yields the upstream velocity of (1.6).

5. Flow at infinity

We obtain the limiting solution as $|x| \to \infty$ with E fixed by letting $\xi \to \pm\infty$ in (3.4 a, b), in which limit the integrals are dominated by the contributions from the neighbourhood of $\beta = 0$. We place the result in the form

$$\boldsymbol{\psi}^{(0)} \sim -\epsilon \boldsymbol{\psi}_1^{(\infty)}(\xi, \eta) \quad (|\xi| \to \infty), \tag{5.1}$$

where

$$\eta = |\xi|^{-\frac{1}{3}} r \equiv (2\Omega/\nu)^{\frac{1}{3}} \tilde{x}^{-\frac{1}{3}} \tilde{r}, \tag{5.2}$$

$\tilde{x}$ and $\tilde{r}$ are the dimensional co-ordinates ($\tilde{x} \equiv Lx$ and $\tilde{r} \equiv Lr$), and

$$\boldsymbol{\psi}_1^{(\infty)} = r \int_0^\infty \{\beta, -\beta^2 \operatorname{sgn} \xi, \beta^3\} \exp(-\beta^3 |\xi|) J_1(\beta r)\, d\beta \tag{5.3a}$$

$$= |\xi|^{-\frac{1}{3}} \eta \int_0^\infty \{s, -\xi^{-\frac{1}{3}} s^2, \xi^{-\frac{2}{3}} s^3\} \exp(-s^3) J_1(s\eta)\, ds. \tag{5.3b}$$

The similarity solution $\boldsymbol{\psi}_1^{(\infty)}$ is equivalent to that obtained by Childress (1964) in his study of slow motion of a sphere in the Stokes limit for a rotating, viscous fluid, $a/E \to 0$, where La is the radius of the sphere. It is a member of the set of similarity solutions,

$$\boldsymbol{\psi}_n^{(\infty)}(\xi, \eta) = |\xi|^{-\frac{1}{3}n} \eta \int_0^\infty \{1, -\xi^{-\frac{1}{3}} s, \xi^{-\frac{2}{3}} s^2\} s^n \exp(-s^3) J_1(s\eta)\, ds, \tag{5.4}$$

each of which satisfies (3.2) as $|\xi| \to \infty$ and represents a balance between viscous and Coriolis forces. The mass flux across a transverse plane ($x = \text{const.}$) associated with $\boldsymbol{\psi}_n^{(\infty)}$ is $2\pi\rho U L^2$ for $n = 0$ and zero for $n > 0$ (it is infinite for $n < 0$). The axial component of the perturbation velocity implied by $\boldsymbol{\psi}_n^{(\infty)}$ is

$$u_n^{(\infty)} = r^{-1} \partial_r \psi_n^{(\infty)} = (E|x|)^{-\frac{1}{3}(n+2)} \mathscr{U}_n(\eta), \tag{5.5}$$

where

$$\mathscr{U}_n(\eta) = \int_0^\infty s^{n+1} \exp(-s^3) J_0(s\eta)\, ds \tag{5.6a}$$

$$= \frac{1}{3} \sum_{m=0}^{\infty} (-)^m (m!)^{-2} \Gamma\left(\frac{2m+n+2}{3}\right) (\tfrac{1}{2}\eta)^{2m} \tag{5.6b}$$

$$\sim \tfrac{1}{3} 2^{\frac{1}{2}} (\tfrac{1}{3}\eta)^{\frac{1}{2}(n-1)} \exp[-(2\eta^3/27)^{\frac{1}{2}}] \cos[(2\eta^3/27)^{\frac{1}{2}} + \tfrac{1}{4}(n-1)\pi] \quad (\eta \to \infty), \tag{5.6c}$$

and (5.6b) and (5.6c) follow from (5.6a) through, respectively, a power-series expansion of the integrand and a saddle-point approximation.

The radial profile $3\mathscr{U}_1(\eta)$ is compared with the radial profile, $J_0(r)$, of (1.6) and (1.7) in figure 1. Observations of the radial profiles of u at different axial stations, both upstream and downstream of a given body, could provide definitive information on the parametric domains of the limiting solutions considered here.

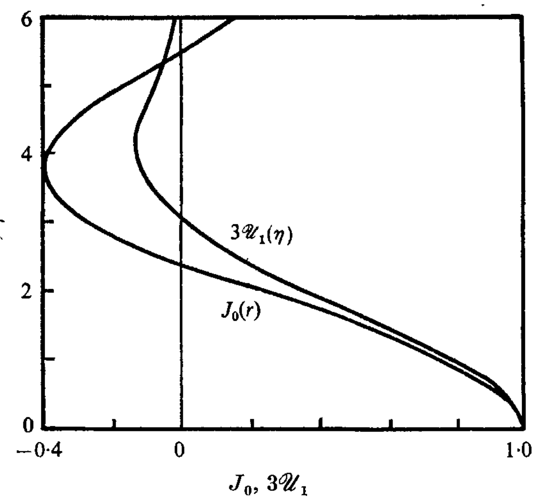

FIGURE 1. The function $3\mathscr{U}_1(\eta)$ of (5.5) and (5.6) compared with $J_0(r)$.

This work was initially undertaken at La Jolla, where it was partially supported by a Grant from the National Science Foundation and by Contract Norn-2216(29) with the Office of Naval Research. It was completed at the University of Cambridge, where it was partially supported by the John Simon Guggenheim Memorial Foundation and by the United States–United Kingdom Educational Commission (under the Fulbright Program). I am indebted to G. K. Batchelor, T. B. Benjamin, F. P. Bretherton, L. E. Fraenkel and K. Stewartson for stimulating discussions.

Appendix. Asymptotic evaluation of $\boldsymbol{\phi}$ as $\xi \downarrow 0$

We require the asymptotic evaluation of $\boldsymbol{\phi}(\xi, n)$, given by (3.9), as $\xi \downarrow 0$. The exponentials in (3.9b) are significant as $\xi \downarrow 0$ only for $\beta^2\xi = O(1)$ and may be approximated by

$$\exp(-\beta^3|\beta \pm 1|^{-1}\xi) \sim \exp(-\beta^2\xi)\,[1 \pm \beta\xi - \xi + \tfrac{1}{2}\beta^2\xi \mp \beta\xi^2 + O(\xi^2)] \quad [\beta^2\xi = O(1),\ \xi \downarrow 0]. \quad \text{(A 1)}$$

Substituting (A 1) and the identity

$$\{1,\ \mp\beta, \beta^2\}(\beta \pm 1)^{-1} = \mathbf{I}(\beta \pm 1)^{-1} + \{0,\ \mp 1, \beta \mp 1\} \quad \text{(A 2)}$$

into (3.9b) and combining the results in (3.9a), we obtain

$$\boldsymbol{\phi} \sim 2r \int_0^\infty \big(\mathbf{I}[(1-2\xi)\beta^2(1-\beta^2)^{-1} - \tfrac{1}{2}\beta^2\xi^2 + O(\xi^2)] + \{0, \beta^2\xi[1-\xi+O(\xi^2)], -\beta^2[1-2\xi+\tfrac{1}{2}\beta^2\xi^2+O(\xi^2)]\}\big)\exp(-\beta^2\xi)\,J_1(\beta r)\,d\beta \quad (\xi \downarrow 0). \quad \text{(A 3)}$$

Separating out the singular integral

$$\phi_1(\xi, r) = 2r \fint_0^\infty \beta^2(1-\beta^2)^{-1} \exp(-\beta^2\xi) J_1(\beta r)\, d\beta \tag{A 4}$$

and invoking the known integral

$$r\int_0^\infty \exp(-\beta^2\xi) J_1(\beta r)\, d\beta = 1 - e^{-\zeta} \quad (\zeta \equiv \tfrac{1}{4}\xi^{-1}r^2), \tag{A 5}$$

we may rewrite (A 3) in the form

$$\boldsymbol{\phi} \sim \mathbf{I}[(1-2\xi)\phi_1 + \xi^2\partial_\xi(1-e^{-\zeta})] + 2\{0, -\xi(1-\xi)\partial_\xi, (1-2\xi)\partial_\xi - \tfrac{1}{2}\xi^2\partial_\xi^2\}(1-e^{-\zeta}). \tag{A 6}$$

Remarking that $\quad (\partial_\xi + 1)\phi_1 = -2\partial_\xi(1-e^{-\zeta}) = \tfrac{1}{2}r^2\xi^{-2}e^{-\zeta}, \quad$ (A 7)

we obtain

$$\phi_1(\xi, r) = \phi_1(0, r) + \tfrac{1}{2}r^2 \int_0^\xi e^{\eta-\xi}\,\eta^{-2} \exp(-\tfrac{1}{4}r^2\eta^{-1})\, d\eta \tag{A 8a}$$

$$= \phi_1(0, r) + 2(1-\xi)\, e^{-\zeta} + \tfrac{1}{2}r^2 E_1(\zeta) + O(\xi^2), \tag{A 8b}$$

where $E_1(\zeta)$ is the exponential integral (3.12). Setting $\xi = 0$ in (A 4) we obtain

$$\phi_1(0, r) = 2r \fint_0^\infty \beta^2(1-\beta^2)^{-1} J_1(\beta r)\, d\beta = -2r\partial_r \fint_0^\infty \beta(1-\beta^2)^{-1} J_0(\beta r)\, d\beta$$

$$= -\pi r \partial_r Y_0(r) = \pi r Y_1(r). \tag{A 9}$$

Substituting (A 8) and (A 9) into (A 6) and carrying out the differentiations with respect to ξ, we obtain

$$\boldsymbol{\phi} \sim \mathbf{I}[(1-2\xi)\,\pi r Y_1(r) + \tfrac{1}{2}r^2 E_1(\zeta) + 2(1-3\xi-\tfrac{1}{8}r^2)\, e^{-\zeta} + O(\xi^2)]$$
$$+ 2\{0, 1 + O(\xi), -\xi^{-1} + 1 + \tfrac{1}{2}\zeta + O(\xi)\}\,\zeta e^{-\zeta}. \tag{A 10}$$

REFERENCES

CHILDRESS, S. 1964 The slow motion of a sphere in a rotating, viscous fluid. *J. Fluid Mech.* **20**, 305–314.

FRAENKEL, L. E. 1956 On the flow of a rotating fluid past bodies in a pipe. *Proc. Roy. Soc.* A **233**, 506–526.

FRAENKEL, L. E. 1969 On the method of matched asymptotic expansions. *Proc. Camb. Phil. Soc.* **65**, 209–231, 233–261, 263–284.

GOLDSTEIN, S. 1938 *Modern Developments in Fluid Dynamics.* Cambridge University Press.

LONG, R. R. 1953 Steady motion around a symmetrical obstacle moving along the axis of a rotating fluid. *J. Meteor.* **10**, 197–203.

MAXWORTHY, T. 1970 The flow created by a sphere moving along the axis of a rotating, slightly viscous fluid. *J. Fluid Mech.* **40**, 453–480.

MILES, J. W. 1969 The lee-wave régime for a slender body in a rotating flow. *J. Fluid Mech.* **36**, 265–288.

MILES, J. W. 1970 The lee-wave régime for a slender body in a rotating flow. Part 2. *J. Fluid Mech.* **42**, 201–206.

STEWARTSON, K. 1968*a* A note on a source in a rotating liquid. *Proc. Camb. Phil. Soc.* **64**, 507–511.

STEWARTSON, K. 1968*b* On inviscid flow of a rotating fluid past an axially-symmetric body using Oseén's equations. *Quart. J. Mech. Appl. Math.* **21**, 353–373.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics.* New York: Academic.